Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

On the linear extension complexity of regular n-gons

Arnaud Vandaele [a,*], Nicolas Gillis [a,1], François Glineur [b,2]

[a] *Department of Mathematics and Operational Research, Faculté Polytechnique,*
Université de Mons, Rue de Houdain 9, 7000 Mons, Belgium
[b] *Université catholique de Louvain, CORE and ICTEAM Institute, B-1348*
Louvain-la-Neuve, Belgium

A B S T R A C T

In this paper, we propose new lower and upper bounds on the linear extension complexity of regular n-gons. Our bounds are based on the equivalence between the computation of (i) an extended formulation of size r of a polytope P, and (ii) a rank-r nonnegative factorization of a slack matrix of the polytope P. The lower bound is based on an improved bound for the rectangle covering number (also known as the boolean rank) of the slack matrix of the n-gons. The upper bound is a slight improvement of the result of Fiorini, Rothvoss and Tiwary (2012) [9]. The difference with their result is twofold: (i) our proof uses a purely algebraic argument while Fiorini et al. used a geometric argument, and (ii) we improve the base case allowing us to reduce their upper bound $2\lceil \log_2(n) \rceil$ by one when $2^{k-1} < n \leq 2^{k-1} + 2^{k-2}$ for some integer k. We conjecture that this new upper bound is tight, which is suggested by numerical experiments for small n. Moreover, this improved upper bound allows us to close the gap with the best known lower bound for certain regular n-gons (namely,

* Corresponding author.
E-mail addresses: arnaud.vandaele@umons.ac.be (A. Vandaele), nicolas.gillis@umons.ac.be (N. Gillis), francois.glineur@uclouvain.be (F. Glineur).

$9 \leq n \leq 13$ and $21 \leq n \leq 24$) hence allowing for the first time to determine their extension complexity.

1. Introduction

An extended formulation (or extension) for a polytope P is a higher dimensional polyhedron Q such that there exists a linear map π with $\pi(Q) = P$. The size of such an extended formulation is defined as the number of facets of the polyhedron Q. The size of the smallest possible extension of P is called the (linear) extension complexity of P and is denoted $\mathrm{xc}(P)$. The quantity $\mathrm{xc}(P)$ is of great importance since it characterizes the minimum information necessary to represent P. In particular, in combinatorial optimization, it characterizes the minimum size necessary to represent a problem as a linear programming problem (taking P as the convex hull of the set of feasible solutions). Hence although P might have exponentially many facets, Q might only have polynomially many, providing a way to solve linear programs over P much more effectively. An example of such a polytope is the permutahedron, that is, the convex hull of all permutations of the set $\{1, 2, \ldots, n\}$ with $n!$ vertices and $2^n - 2$ facet-defining inequalities, that can be represented as the projection of a polyhedron with $\mathcal{O}(n \log(n))$ facets [12].

The characterization of the extension complexity has attracted much interest recently; in particular lower bounds have been investigated since they provide provable limits of linear programming to solve combinatorial optimization problems; see, e.g., [8]. For example, it was recently shown that the extension complexity of the matching polytope is exponential (in the number of vertices of the graph), answering a long-standing open question about the existence of a polynomial-size linear programming formulation for the matching problem [20], which implies that although it is solvable in polynomial time, the standard formulation cannot be written as a linear program with a polynomial number of inequalities.

Interestingly, most lower bounds for the extension complexity of polytopes are based on a well-known linear algebra concept: the nonnegative rank. The nonnegative rank of a nonnegative m-by-n matrix M, denoted $\mathrm{rank}_+(M)$, is the minimum r such that there exist a nonnegative m-by-r matrix U and a nonnegative r-by-n matrix V such that $M = UV$. The pair (U, V) is a rank-r nonnegative factorization of M. The link between the nonnegative rank and the extension complexity of a polytope, a seminal result of Yannakakis [24], goes as follows. Let P be a polytope in dimension d with

- f facets expressed as linear inequalities $a_i^T x \leq b_i$, $1 \leq i \leq f$, and
- v vertices denoted $x_j \in \mathbb{R}^d$, $1 \leq j \leq v$.

The slack matrix $S_P \in \mathbb{R}_+^{f \times v}$ of P is defined as

$$S_P(i,j) = b_i - a_i^T x_j \geq 0, \qquad \text{for all } 1 \leq i \leq f, 1 \leq j \leq v.$$

Note that the slack matrix of a polytope is not unique since the inequalities can be scaled, and the rows and columns permuted, but this does not influence its nonnegative rank; see [13] for more details. Note also that $\text{rank}(S_P) = d + 1$ if P is full dimensional. Then, we have according to [24] that

$$\text{rank}_+(S_P) = \text{xc}(P).$$

Moreover any nonnegative factorization $(U, V) \geq 0$ of $S_P = UV$ provides an explicit extended formulation for P (with some redundant equalities):

$$P = \{x \in \mathbb{R}^d \mid Ax \leq b\} = \{x \in \mathbb{R}^d \mid Ax + Uy = b \text{ and } y \geq 0\},$$

where $A \in \mathbb{R}^{f \times d}$ with $A(i, :) = a_i$ for all i, and $b \in \mathbb{R}^f$. For example, the matrix

$$S_6 = \begin{pmatrix} 0 & 1 & 2 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 2 & 1 \\ 1 & 0 & 0 & 1 & 2 & 2 \\ 2 & 1 & 0 & 0 & 1 & 2 \\ 2 & 2 & 1 & 0 & 0 & 1 \\ 1 & 2 & 2 & 1 & 0 & 0 \end{pmatrix}$$

is a slack matrix of the regular hexagon (hence it has rank three) and has nonnegative rank equal to five:

$$S_6 = \begin{pmatrix} 0 & 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{pmatrix}.$$

This implies that the regular hexagon can be described as the projection of a higher dimensional polytope with 5 facets; see Fig. 1 for an illustration. In this paper, we focus on the extension complexity of regular n-gons, and in particular on a new upper bound.

Extension complexity of regular n-gons   In the remainder of this paper, we denote S_n the slack matrix of the regular n-gon (more precisely, any slack matrix; see Section 2 for a construction), hence $\text{rank}_+(S_n)$ equals the extension complexity of the regular n-gon; see above. In the following, we describe several bounds for the nonnegative rank, focusing on the slack matrices of regular n-gons.
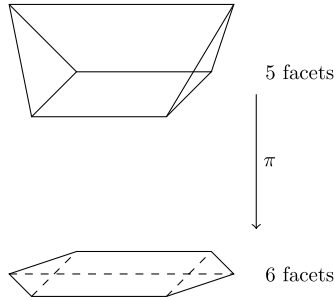
Fig. 1. Minimum-size extension of the regular hexagon.

Lower bounds. There exist several approaches to derive lower bounds for the nonnegative rank, which we classify in three classes:

- *Geometric.* Using a counting argument and the facts that (i) any face of a polytope is the projection of a face of its extension, and (ii) any face is an intersection of facets, it can be shown that $\operatorname{rank}_+(S_n) \geq \lceil \log_2(2n+2) \rceil$ [12]. Based on a refined geometric counting argument, Gillis and Glineur [10] described a stronger lower bound for the slack matrix of polygons[3]: the nonnegative rank $r_+ = \operatorname{rank}_+(S_n)$ of S_n must satisfy

$$n \leq \max_{3 \leq d \leq r_+ - 1} \ \min_{i=0,1} \ \operatorname{faces}(r_+, d-1, d-3+i),$$

  where the quantity $\operatorname{faces}(v, d, k)$ is the maximal number of k-faces of a polytope with v vertices in dimension d, attained by cyclic polytopes [18]; see also [25, p. 257, Corollary 8.28]. We have

$$\operatorname{faces}(v, d, k-1) = \sum_{i=0}^{\frac{d}{2}} {}^{*}\left(\binom{d-i}{k-i} + \binom{i}{k-d+i} \right) \binom{v-d-1+i}{i},$$

  where $\sum^{*}$ denotes a sum where only half of the last term is taken for $i = \frac{d}{2}$ if d is even, and the whole last term is taken for $i = \lfloor \frac{d}{2} \rfloor = \frac{d-1}{2}$ if d is odd. This bound can be generalized to any nonnegative matrix [10], but it becomes difficult to compute for non-slack matrices as it requires computing the restricted nonnegative rank, which is in general NP-hard (but is simply equal to the number of vertices for a slack matrix).

- *Combinatorial.* These bounds are based on the sparsity pattern of the input matrix. The most well-known one is the rectangle covering bound (RCB) that counts the minimum number of rectangles necessary to cover all positive entries of the matrix, a rectangle being a subset of rows and columns for which the corresponding submatrix contains only positive entries; see [7] and the references therein. Note that the RCB

---

[3] They actually derived this bound for linear Euclidean distance matrices, but it also applies to the slack matrix of polygons.

is equal to the boolean rank; see, e.g., [4]. A closely related bound is the refined rectangle covering bound (RRCB) by Oelze, Vandaele, Weltge [19]: in addition to covering every positive entry by a rectangle, the RRCB requires that every 2-by-2 nonsingular submatrix is touched by at least two rectangles (note that the same rectangle can be used twice). For example, the RCB for the matrix

$$S_9 = \begin{pmatrix} 1 & 2 & 0 & 3 \\ 4 & 5 & 6 & 0 \\ 7 & 8 & 9 & 0 \end{pmatrix}$$

is equal to two while the RRCB is equal to three. In fact, this matrix contains only three maximal rectangles (that is, rectangles not contained in any larger rectangle):

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \text{ and } \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix},$$

and only two of them are required to cover all positive entries (the last two, which is the unique solution) while three are necessary to touching twice all rank-two positive submatrices (which is tight since this is a 3-by-4 matrix), e.g., the block $\begin{pmatrix} 4 & 5 \\ 7 & 8 \end{pmatrix}$ touched only once with the RCB solution.

Although these bounds can be rather strong in some cases, they are computationally very expensive, and only work well for matrices with 'well located' zero entries. For the slack matrices of the regular n-gons, we could compute them up to $n = 13$ (for larger n, it would take several weeks of computation with our current formulation).

- *Convex Relaxations.* Fawzi and Parrilo developed two lower bounds for the nonnegative rank based on a sum-of-squares approximation of the copositive cone [5,6]. These bounds are very general as they can be computed for any nonnegative matrix; however they are typically weaker than the aforementioned lower bounds, in particular for slack matrices.

These bounds are compared for the regular n-gons in Fig. 2. We observe that the best lower bounds are the geometric bound from [10] and the rectangle covering bounds [7,19] that coincide except for $n = 9, 13$ for which only the RRCB is tight (as it matches the best upper bound; see below).

Upper bounds. Ben-Tal and Nemirovski [3] gave an extension of the regular n-gons when n is a power of two ($n = 2^k$ for some k) with $2 \log_2(n) + 4$ facets. They used this construction to approximate the circle with regular n-gons which allowed them to approximate second-order cone programs with linear programs. This construction was slightly reduced to size $2 \log_2(n)$ in [11] (again, only for $n = 2^k$). Kaibel and Pashkovich [16,15] proposed a general construction for arbitrary n of size $2 \lceil \log_2(n) \rceil + 2$. Fiorini, Rothvoss and Tiwary [9] improved the bound to $2 \lceil \log_2(n) \rceil$, which is, to the best of our knowledge, the best known upper bound for regular n-gons. These last bounds are based on
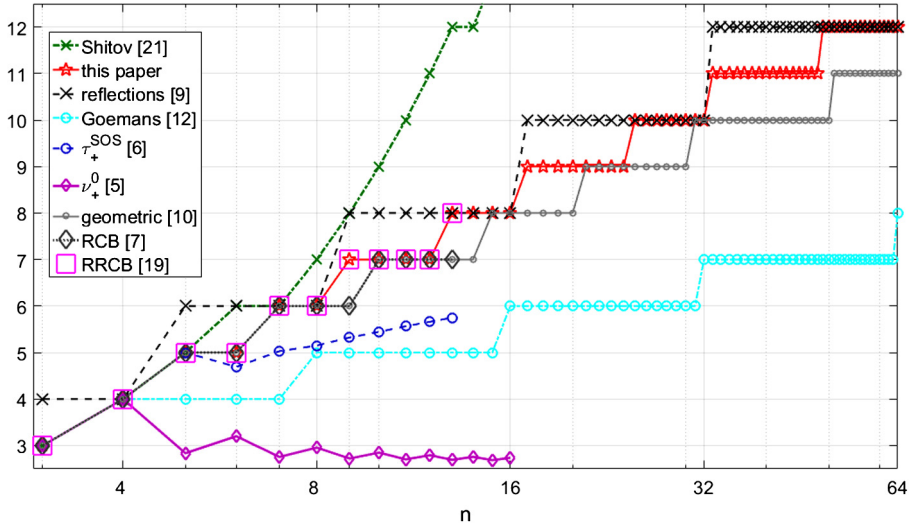
Fig. 2. Comparison of lower and upper bounds for the nonnegative rank of the slack matrices of regular n-gons, that is, $\text{rank}_+(S_n)$. (Note that some bounds cannot be computed for all n because of their high computational cost.)

Table 1
Comparison of two lower bounds (first two rows) and our new upper bound (see Equation (1) on page 223) for the nonnegative rank of regular n-gons. Bold indicates the tight bounds, that is, bounds that coincide with the nonnegative rank.

n	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
RRCB [19]	**5**	**6**	**6**	**7**	7	7	7	**8**	?	?	?	?	?	?	?	?
geometric [10]	**5**	**6**	**6**	6	7	7	7	7	7	**8**	**8**	8	8	8	8	**9**
Equation (1)	**5**	**6**	**6**	**7**	7	7	7	**8**	8	**8**	**8**	9	9	9	9	**9**

a geometric argument using successive reflections to construct the regular n-gon. Note that Shitov [21] proved an upper bound of $\lceil \frac{6n}{7} \rceil$ for the nonnegative rank of any n-by-n rank-three nonnegative matrix, hence also valid for the slack matrix of polygons.

As shown in Fig. 2, prior to our new upper bound, the exact value of $\text{rank}_+(S_n)$ is not known for most values of n larger than 9, as the best lower and upper bounds do not coincide. Therefore, the exact value of the extension complexity of many regular n-gons is still unknown.

Table 1 also gives the best upper and lower bounds for n up to 21.

Contribution of the paper   In this paper, our contribution is mainly twofold. First, in Section 3, we derive an improved lower bound for the rectangle covering number r of the slack matrix of regular n-gons. We show that the following relation holds

$$n \leq \frac{r - \lfloor r/2 \rfloor}{r - 1} \binom{r}{\lfloor r/2 \rfloor},$$

which improves over the best previously known relation given by $n \leq \binom{r}{\lfloor r/2 \rfloor}$ (see [4]). Although this new lower bound does not improve the best known lower bounds for the nonnegative rank of the slack matrices of regular n-gons (namely, the RRCB and the geometric bound; see previous paragraph), it is applicable to a broader class of matrices, namely those which have the same sparsity pattern as the slack matrices of n-gons. Moreover, it turns out to be a tight bound for the rectangle covering number, a.k.a. the boolean rank, for some n (comparing it with the upper bound from [1]).

Second, we slightly improve the upper bound of Fiorini, Rothvoss and Tiwary [9]. Although our approach is similar to that of Fiorini et al., both being recursive, our proof is rather different, being purely algebraic as opposed to their geometric approach. Moreover, we are able to reduce the upper bound by one when $2^{k-1} < n \leq 2^{k-1} + 2^{k-2}$ for some k: this is possible by stopping the recursion earlier at a better base case (note that it would be possible to modify the proof of Fiorini et al. to achieve the same bound). We show that for all $n \geq 2$,

$$\text{rank}_+(S_n) \leq \begin{cases} 2\lceil \log_2(n) \rceil - 1 = 2k - 1 & \text{for} & 2^{k-1} & < & n & \leq 2^{k-1} + 2^{k-2}, \\ 2\lceil \log_2(n) \rceil = 2k & \text{for} & 2^{k-1} + 2^{k-2} & < & n & \leq & 2^k. \end{cases} \tag{1}$$

Although the improvement is relatively minor, our numerical experiments strongly suggest that this bound is tight; see the discussion at the end of Section 4. Moreover, our bound allows us to close the gap for several n-gons as it matches the best known lower bound. Indeed, our bound implies that $\text{rank}_+(S_n) = 7$ for $9 \leq n \leq 12$ and that $\text{rank}_+(S_n) = 9$ for $21 \leq n \leq 24$; see Fig. 2. (Note that the RRCB was, to the best of our knowledge, never computed for $n = 13$ prior to this work, hence it is also the first time $\text{rank}_+(S_n) = 8$ is claimed for $n = 13$.)

The paper is organized as follows. In Section 2, we briefly describe the construction of the slack matrices of regular n-gons. In Section 3, we describe our new improved lower bound for the rectangle covering number of these matrices, and, in Section 4, we describe our construction that proves the aforementioned upper bound. Then we discuss some directions for further research and conclude in Section 5.

2. The slack matrices of regular n-gons

Let us construct the slack matrices of regular n-gons. Without loss of generality (w.l.o.g.), we use regular n-gons centered at the origin with their vertices located on the unit circle; see Fig. 3 for an illustration with the pentagon. The length s of the facets of the regular n-gon is given by $s = 2\sin\left(\frac{\pi}{n}\right)$. The slack between a facet and the kth vertex (the 0th and (n-1)th being on the considered facet, and counting along the circle in any direction) is equal to:

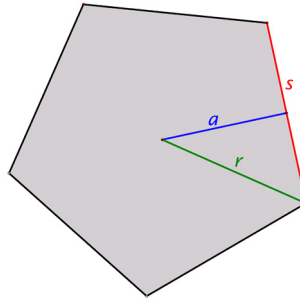$$c_k = \cos\left(\frac{\pi}{n}\right) - \cos\left((2k+1)\frac{\pi}{n}\right). \tag{2}$$

Fig. 3. Illustration for the construction of the slack matrices of regular n-gons.

By symmetry, (i) our slack matrices of regular n-gons are circulant matrices for which the vector c is translated one element to the right on each row, and (ii) the vector c satisfies $c_k = c_{n-1-k}$ for all k. For example, for $n = 9$, we have

$$S_9 = \begin{pmatrix} 0 & c_1 & c_2 & c_3 & c_4 & c_3 & c_2 & c_1 & 0 \\ 0 & 0 & c_1 & c_2 & c_3 & c_4 & c_3 & c_2 & c_1 \\ c_1 & 0 & 0 & c_1 & c_2 & c_3 & c_4 & c_3 & c_2 \\ c_2 & c_1 & 0 & 0 & c_1 & c_2 & c_3 & c_4 & c_3 \\ c_3 & c_2 & c_1 & 0 & 0 & c_1 & c_2 & c_3 & c_4 \\ c_4 & c_3 & c_2 & c_1 & 0 & 0 & c_1 & c_2 & c_3 \\ c_3 & c_4 & c_3 & c_2 & c_1 & 0 & 0 & c_1 & c_2 \\ c_2 & c_3 & c_4 & c_3 & c_2 & c_1 & 0 & 0 & c_1 \\ c_1 & c_2 & c_3 & c_4 & c_3 & c_2 & c_1 & 0 & 0 \end{pmatrix}. \tag{3}$$

Note that, to the best of our knowledge, the best known lower (resp. upper) bound for $\mathrm{rank}_+(S_9)$ is 7 (resp. 8). In this paper, we will improve the upper bound to 7 hence proving that $\mathrm{rank}_+(S_9) = 7$; see Fig. 2.

3. Lower bound for the boolean rank of S_n

In this section, we improve the lower bound on the boolean rank (or, equivalently, the rectangle covering number) for regular n-gons. On the way, we derive several new interesting results that could be used to derive other bounds.

Given a nonnegative m-by-n matrix M, let $U, V \geq 0$ be an exact nonnegative factorization of $M = UV$ of size r. In this section, we will use the following notation. Let us define the following subsets of $\{1, 2, \ldots r\}$, representing the supports of the rows of U and columns of V:

$$s_i = \{k \mid U_{ik} \neq 0\}, \ 1 \leq i \leq m \quad \text{and} \quad t_j = \{k \mid V_{kj} \neq 0\}, \ 1 \leq j \leq n.$$

Since $M_{ij} = U(i, :)V(:, j)$, $U \geq 0$ and $V \geq 0$, we have

$$M_{ij} = 0 \quad \Longleftrightarrow \quad s_i \cap t_j = \emptyset. \tag{4}$$

If $s_i \subseteq s_l$ for some i, l, (4) implies that the sparsity pattern of the ith row of M is contained in the sparsity pattern of the lth row of M (and similarly for the columns). Therefore, if M contains p rows whose sparsity patterns are not contained in one another, there exist p subsets from s_i $(1 \leq i \leq n)$ that form a Sperner family of size p, also known as an antichain of size p, which is a family of p sets that are not contained in one another [22]. By symmetry, the same holds for the columns.

3.1. Sperner's theorem and rectangle covering

Sperner's theorem bounds the size of an antichain over r elements. Let us recall this result and a proof that will be useful later.

Theorem 1. *Let $\mathcal{S} = \{s_1, s_2, \ldots, s_n\}$ be a set of n subsets of $\{1, 2, \ldots, r\}$. Suppose $\mathcal{S}$ is an antichain, that is, no subset in $\mathcal{S}$ is contained in another subset in $\mathcal{S}$. Then,*

$$n \leq \binom{r}{\lfloor r/2 \rfloor}, \tag{5}$$

and the bound is tight (take all subsets of size $\lfloor r/2 \rfloor$).

Proof. ([17]) This proof is based on a counting argument using the fact that there are $r!$ permutations of $\{1, 2, \ldots, r\}$. Given $s_i \in \mathcal{S}$ with k elements, there are $k!(r - k)!$ permutations of $\{1, 2, \ldots, r\}$ whose first k elements are in s_i. Because the s_i's are not contained in one another, the permutations generated for two different subsets s_i and s_j cannot coincide (otherwise this would imply that $s_i \subset s_j$ or $s_j \subset s_i$). Let us also denote a_k the number of sets with k elements contained in $\mathcal{S}$, that is, $a_k = |\{s \in S \mid |s| = k\}|$, hence $n = \sum_{k=0}^{r} a_k$. We have

$$\sum_{k=0}^{r} a_k k!(r - k)! \quad \leq \quad r! \,.$$

Therefore,

$$\frac{n}{\binom{r}{\lfloor r/2 \rfloor}} = \sum_{k=0}^{r} \frac{a_k}{\binom{r}{\lfloor r/2 \rfloor}} \quad \leq \quad \sum_{k=0}^{r} \frac{a_k}{\binom{r}{k}} = \sum_{k=0}^{r} a_k \frac{k!(r - k)!}{r!} \quad \leq \quad 1,$$

since $\binom{r}{\lfloor r/2 \rfloor} \geq \binom{r}{k}$ for all k. This completes the proof. $\quad\square$

The above result was used to prove that the rectangle covering number of the n-by-n Euclidean distance matrices (with zeros only the diagonal) is the minimum r such that $n \leq \binom{r}{\lfloor r/2 \rfloor}$; see [2] and the references therein. This result can actually be generalized for any nonnegative matrix.

Corollary 1 *([4]). Let M be a matrix having p rows or p columns whose sparsity patterns are not contained in one another. Then,*

$$RCB(M) \geq \min\left\{ r \middle| \binom{r}{\lfloor r/2 \rfloor} \geq p \right\}.$$

Proof. Let M have p rows with different sparsity patterns. As explained in the introduction of this section, this implies that there are p subsets of $\{1, 2, \ldots, r\}$ corresponding to the sparsity patterns of p rows of U that are not contained in one another. Theorem 1 allows to conclude.   □

In particular, this result can be applied to the slack matrix of any polytope. In fact, the slack of two different vertices cannot be contained in one another, otherwise it would mean that a vertex is the intersection of a subset of the facets intersecting at another vertex. The same holds for two different facets by polar duality or a similar argument.

Corollary 2. *Let M be the slack matrix of a polytope with f facets and v vertices. Then,*

$$RCB(M) \geq \min\left\{ r \middle| \binom{r}{\lfloor r/2 \rfloor} \geq \max(f, v) \right\}.$$

Note that, the results from Corollaries 1 and 2 were already known prior to this work; see, e.g., [14, Cor. 4.13] for a more general result.

In the next section, we apply the same ideas to improve the lower bound for the rectangle covering number of the slack matrices of n-gons.

3.2. Improvement for n-gons

Let M be the slack matrix of a n-gon such that $M_{ij} = 0$ if and only if $i = j$ or $i = (j + 1) \bmod n$ for $1 \leq i, j \leq n$; see Section 2. To simplify the heavy notation mod n, we will assume throughout this section that $i = 1 \equiv n + 1$ when i represents an index. As before, let $UV = M$ be a nonnegative factorization of size r of M, let s_i denote the support of the ith row of U ($1 \leq i \leq n$) and t_j the support of the jth column of V ($1 \leq j \leq n$). We have $M_{ij} = 0$ if and only if $i = j$ or $i = j + 1$, and

$$M_{ij} = 0 \quad \Longleftrightarrow \quad s_i \cap t_j = \emptyset.$$

Let us try to characterize the size of the sets $\mathcal{S} = \{s_1, s_2, \ldots, s_n\}$ and $\mathcal{T} = \{\overline{t_1}, \overline{t_2}, \ldots, \overline{t_n}\}$ that satisfy the above property, where $\overline{a}$ denotes the complement of set a.

First, we can assume without loss of generality that $t_i = \overline{s_i \cup s_{i+1}}$. In fact, $t_i = \overline{s_i \cup s_{i+1}}$ is the largest possible set that does not intersect $s_i \cup s_{i+1}$ while having the most intersections with all other sets in $\mathcal{S}$ (which is the best possible situation since $M_{ij} > 0$ for $i \neq j, j + 1$).

For the same reason as in Corollary 1, since the rows and columns of M have different sparsity patterns, we have that

(C1) $\mathcal{S} = \{s_1, s_2, \ldots, s_n\}$ is an antichain.

(C2) $\mathcal{T} = \{s_1 \cup s_2, s_2 \cup s_3, \ldots, s_{n-1} \cup s_n, s_n \cup s_1\}$ is an antichain, since taking the complement of all the sets in an antichain gives another antichain of the same size.

(C3) Every set $s_i \subseteq \{1, 2, \ldots, r\}$ contains at least one element not in the sets $\overline{t_j} = s_j \cup s_{j+1}$ for $j, j+1 \neq i$, since $M_{ij} > 0$ for $i \neq j, j+1$.

Theorem 2. *Let $\mathcal{S}$ and $\mathcal{T}$ satisfy (C1–C3) and $r \geq 2$. Then*

$$n \leq \frac{r - \lfloor r/2 \rfloor}{r - 1} \binom{r}{\lfloor r/2 \rfloor}.$$

Proof. Let us denote k_i the number of elements in s_i, z_i the number of additional elements in $\overline{t_i}$ compared to s_i (that is, $|\overline{t_i}| = k_i + z_i$) and z_i' the number of additional elements in $\overline{t_{i-1}}$ compared to s_i (that is, $|\overline{t_{i-1}}| = k_i + z_i'$). Following the same argument as in Theorem 1, we have that the number of permutations with the elements of s_i in the first positions is given by $k_i!(r - k_i)!$, of $\overline{t_i}$ by $(k_i + z_i)!(r - k_i - z_i)!$, and of $\overline{t_{i-1}}$ by $(k_i + z_i')!(r - k_i - z_i')!$. However, between s_i and $\overline{t_i}$, there are $k_i!z_i!(r - k_i - z_i)!$ common permutations (and similarly between s_i and $\overline{t_{i-1}}$). Note that these are the only possible repetitions because of (C3). Note also that $|\overline{t_i}| = k_i + z_i = k_{i+1} + z_{i+1}'$ hence the number of permutations corresponding to $\overline{t_i}$ are also equal to $1/2(k_i!z_i!(r - k_i - z_i)! + k_{i+1}!z_{i+1}!(r - k_{i+1} - z_{i+1}))$. Counting all permutations corresponding to s_i and $\overline{t_i}$ for $1 \leq i \leq n$ and accounting for the repetitions, we get

$$\sum_{i=1}^{n} k_i!(r - k_i)! + \frac{1}{2}(k_i + z_i)!(r - k_i - z_i)! + \frac{1}{2}(k_i + z_i')!(r - k_i - z_i')!$$

$$- k_i!z_i!(r - k_i - z_i)! - k_i!z_i'!(r - k_i - z_i')! \leq r!.$$

Let us find a lower bound for the left hand side of the above inequality. To do so, we minimize over each term of the sum independently. Noting that z_i and z_i' have exactly the same role, we can assume without loss of generality that $z_i = z_i'$ at a minimum. Removing the index i for simplicity, we therefore have to evaluate

$$\min_{k \geq 1, z \geq 1, k+z \leq r} k!(r - k)! + (k + z)!(r - k - z)! - 2k!z!(r - k - z)!.$$

In Appendix A, we show that $k^* = \lfloor r/2 \rfloor$ and $z^* = 1$ is an optimal solution. Therefore, dividing the inequality above by $r!$ and using our lower bound for each term (replacing the k_i's with $\lfloor r/2 \rfloor$ and the z_i's with 1), we obtain

$$n\left(\binom{r}{\lfloor r/2\rfloor}^{-1} + \underbrace{\binom{r}{\lfloor r/2\rfloor}^{-1}\frac{\lfloor r/2\rfloor+1}{r-\lfloor r/2\rfloor}}_{\binom{r}{\lfloor r/2\rfloor+1}^{-1}}\left(1 - 2\frac{1}{\lfloor r/2\rfloor+1}\right)\right) \le 1,$$

from which we get, after simplifications, $n \le \frac{r-\lfloor r/2\rfloor}{r-1}\binom{r}{\lfloor r/2\rfloor}$. $\square$

Corollary 3. *Let r be the rectangle covering number of the slack matrix of any n-gon for $n \ge 2$, then*

$$n \le \frac{r-\lfloor r/2\rfloor}{r-1}\binom{r}{\lfloor r/2\rfloor}.$$

Note that the term $\frac{r-\lfloor r/2\rfloor}{r-1}$ tends to $\frac{1}{2}$ when r grows, and we cannot hope to obtain a better bound using our counting argument. In fact, this is the case when there would be no repetitions between the permutations generated from the sets in $\mathcal{S}$ and $\mathcal{T}$; see the proof of Theorem 2.

The bound from the corollary above also applies to the so-called boolean rank, which is the same as the rectangle covering number. Comparing our bound with the upper bounds computed in [1, p. 145] for small n, our bound is tight for $n = 2-6, 8-9, 13-21, 24-32$ ($a - b$ means from a to b, that is, $a, a+1, \ldots, b$), which was not the case of the previous bound (5) which is tight only for $n = 2 - 4$.

4. Explicit nonnegative factorization of slack matrices S_n of regular n-gons

In this section, we construct a nonnegative factorization of S_n in a recursive way. The idea is the following. At the first step, a rank-two modification of S_n is performed so that the pattern of zero entries of the resulting matrix looks like a cross (see below for an example on S_9). This subdivides the matrix into four blocks exhibiting a lot of symmetry, which can be shown to imply that the nonnegative rank of one block is equal to the nonnegative rank of the full matrix. Then, the same scheme is applied to that subblock until the number of columns of the obtained block B is smaller than four, which we factorize with a trivial decomposition $B = BI$ (I being the identity matrix of appropriate dimension).

Before we rigorously prove that our construction works for any n-gon, let us illustrate the idea on the slack matrix of the regular 9-gon from (3). Observe that the entries of the slack matrix on the main diagonal and the diagonal below it are equal to zero. The first step of our construction will make a rank-two correction of the slack matrix so that the same pattern appears: we remove a matrix from the 4-by-4 lower left block of S_9

$$\begin{pmatrix} c_4 & c_3 & c_2 & c_1 \\ c_3 & c_4 & c_3 & c_2 \\ c_2 & c_3 & c_4 & c_3 \\ c_1 & c_2 & c_3 & c_4 \end{pmatrix} - \begin{pmatrix} c_4-c_3 & c_3-c_2 & c_2-c_1 & c_1 \\ c_3-c_2 & c_4-c_1 & c_3 & c_2 \\ c_2-c_1 & c_3 & c_4 & c_3-c_1 \\ c_1 & c_2 & c_3-c_1 & c_4-c_2 \end{pmatrix} = \begin{pmatrix} c_3 & c_2 & c_1 & 0 \\ c_2 & c_1 & 0 & 0 \\ c_1 & 0 & 0 & c_1 \\ 0 & 0 & c_1 & c_2 \end{pmatrix},$$

and another matrix from the positive 4-by-4 block of S_9 at the upper right (rows 2 to 5, last 4 columns)

$$
\begin{pmatrix}
c_4 & c_3 & \underline{c_2} & \underline{c_1} \\
c_3 & c_4 & c_3 & c_2 \\
\underline{c_2} & c_3 & c_4 & c_3 \\
\underline{c_1} & c_2 & c_3 & c_4
\end{pmatrix}
-
\begin{pmatrix}
c_4 - c_2 & c_3 - c_1 & \underline{c_2} & \underline{c_1} \\
c_3 - c_1 & c_4 & c_3 & c_2 - c_1 \\
\underline{c_2} & c_3 & c_4 - c_1 & c_3 - c_2 \\
\underline{c_1} & c_2 - c_1 & c_3 - c_2 & c_4 - c_3
\end{pmatrix}
=
\begin{pmatrix}
c_2 & c_1 & 0 & 0 \\
c_1 & 0 & 0 & c_1 \\
0 & 0 & c_1 & c_2 \\
0 & c_1 & c_2 & c_3
\end{pmatrix}.
$$

Clearly, the removed matrices are nonnegative since $0 \le c_{k-1} \le c_k$ for all $0 \le k \le \lfloor \frac{n}{2} \rfloor$. Moreover, we show in the next lemma that they have rank one.

Lemma 1. *The (infinite) matrix*

$$
\left[c_{\alpha - i + j} - c_{\beta - i - j} \right]_{i \in \mathbb{Z}, j \in \mathbb{Z}}
$$

has rank one for any fixed $\alpha \in \mathbb{Z}$, $\beta \in \mathbb{Z}$ and $n \in \mathbb{N}_{>0}$.

Proof. We have that $c_k = \cos(\frac{\pi}{n}) - \cos((2k+1)\frac{\pi}{n}) = 2\sin(k\frac{\pi}{n})\sin((k+1)\frac{\pi}{n})$. Choosing any 2×2 minor with rows $i \in \{0, x\}$ and columns $j \in \{0, y\}$ (w.l.o.g.), one can check, using algebra with a few trigonometric identities, that the determinant of

$$
\begin{pmatrix}
c_\alpha - c_\beta & c_{\alpha+y} - c_{\beta-y} \\
c_{\alpha-x} - c_{\beta-x} & c_{\alpha-x+y} - c_{\beta-x-y}
\end{pmatrix}
$$

is equal to zero for any x, y, and any n. $\quad\square$

After these two nonnegative rank-one factors are removed, we obtain

$$
S_9 -
\begin{pmatrix}
0 \\ 0 \\ 0 \\ 0 \\ 0 \\ c_1 \\ c_2 \\ c_3 - c_1 \\ c_4 - c_2
\end{pmatrix}
\begin{pmatrix}
\frac{c_4 - c_3}{c_1} \\ \frac{c_3 - c_2}{c_1} \\ \frac{c_2 - c_1}{c_1} \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0
\end{pmatrix}^T
-
\begin{pmatrix}
0 \\ c_4 - c_2 \\ c_3 - c_1 \\ c_2 \\ c_1 \\ 0 \\ 0 \\ 0 \\ 0
\end{pmatrix}
\begin{pmatrix}
0 \\ 0 \\ 0 \\ 0 \\ 1 \\ \frac{c_2 - c_1}{c_1} \\ \frac{c_3 - c_2}{c_1} \\ \frac{c_4 - c_3}{c_1}
\end{pmatrix}^T
=
\left(
\begin{array}{ccccc|cccc}
0 & c_1 & c_2 & c_3 & c_4 & c_3 & c_2 & c_1 & 0 \\
0 & 0 & c_1 & c_2 & c_3 & c_2 & c_1 & 0 & 0 \\
c_1 & 0 & 0 & c_1 & c_2 & c_1 & 0 & 0 & c_1 \\
c_2 & c_1 & 0 & 0 & c_1 & 0 & 0 & c_1 & c_2 \\
\hline
c_3 & c_2 & c_1 & 0 & 0 & 0 & c_1 & c_2 & c_3 \\
c_3 & c_2 & c_1 & 0 & 0 & 0 & c_1 & c_2 & c_3 \\
c_2 & c_1 & 0 & 0 & c_1 & 0 & 0 & c_1 & c_2 \\
c_1 & 0 & 0 & c_1 & c_2 & c_1 & 0 & 0 & c_1 \\
0 & 0 & c_1 & c_2 & c_3 & c_2 & c_1 & 0 & 0
\end{array}
\right),
$$

with a pattern of zeros forming a cross. This matrix is highly symmetric and has a lot of redundancy: the last four columns (resp. rows) are copies of the first four. Therefore, if we had a nonnegative factorization of the 5-by-5 upper left block, then we would have a nonnegative factorization of the entire matrix with the same nonnegative rank.

To construct that factorization, we apply our strategy recursively: use a rank-two correction to the upper left block to make a cross of zeros appear:

$$
\begin{pmatrix}
0 & c_1 & c_2 & c_3 & c_4 \\
0 & 0 & c_1 & c_2 & c_3 \\
c_1 & 0 & 0 & c_1 & c_2 \\
c_2 & c_1 & 0 & 0 & c_1 \\
c_3 & c_2 & c_1 & 0 & 0
\end{pmatrix}
\rightarrow
\left(\begin{array}{ccc|cc}
0 & c_1 & c_2 & c_1 & 0 \\
0 & 0 & c_1 & 0 & 0 \\
\hline
c_1 & 0 & 0 & 0 & c_1 \\
c_1 & 0 & 0 & 0 & c_1 \\
0 & 0 & c_1 & 0 & 0
\end{array}\right)
=
\left(\begin{array}{ccc}
0 & c_1 & c_2 \\
0 & 0 & c_1 \\
c_1 & 0 & 0 \\
c_1 & 0 & 0 \\
0 & 0 & c_1
\end{array}\right)
\left(\begin{array}{ccc|cc}
1 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0
\end{array}\right).
$$

Now, the upper left block has a trivial nonnegative factorization (since it is a 3-by-3 matrix of rank 3) from which we can eventually derive the following nonnegative factorization for the full matrix S_9:

$$
\begin{pmatrix}
0 & c_1 & c_2 & 0 & c_3-c_1 & 0 & 0 \\
0 & 0 & c_1 & 0 & c_2 & 0 & c_4-c_2 \\
c_1 & 0 & 0 & 0 & c_1 & 0 & c_3-c_1 \\
c_1 & 0 & 0 & c_1 & 0 & 0 & c_2 \\
0 & 0 & c_1 & c_2 & 0 & 0 & c_1 \\
0 & 0 & c_1 & c_2 & 0 & c_1 & 0 \\
c_1 & 0 & 0 & c_1 & 0 & c_2 & 0 \\
c_1 & 0 & 0 & 0 & c_1 & c_3-c_1 & 0 \\
0 & 0 & c_1 & 0 & c_2 & c_4-c_2 & 0
\end{pmatrix}
$$

$$
\times
\begin{pmatrix}
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
\frac{c_2-c_1}{c_1} & 1 & 0 & 0 & 0 & 0 & 0 & 1 & \frac{c_2-c_1}{c_1} \\
0 & 0 & 0 & 1 & \frac{c_2-c_1}{c_1} & 1 & 0 & 0 & 0 \\
\frac{c_4-c_3}{c_1} & \frac{c_3-c_2}{c_1} & \frac{c_2-c_1}{c_1} & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & \frac{c_2-c_1}{c_1} & \frac{c_3-c_2}{c_1} & \frac{c_4-c_3}{c_1}
\end{pmatrix}.
$$

Remark 1. Once the first two rank-one factors have been removed from S_9, the 5-by-5 block could also directly be trivially factorized, and we would obtain

$$
S_9 =
\begin{pmatrix}
0 & c_1 & c_2 & c_3 & c_4 & 0 & 0 \\
0 & 0 & c_1 & c_2 & c_3 & 0 & c_4-c_2 \\
c_1 & 0 & 0 & c_1 & c_2 & 0 & c_3-c_1 \\
c_2 & c_1 & 0 & 0 & c_1 & 0 & c_2 \\
c_3 & c_2 & c_1 & 0 & 0 & 0 & c_1 \\
c_3 & c_2 & c_1 & 0 & 0 & c_1 & 0 \\
c_2 & c_1 & 0 & 0 & c_1 & c_2 & 0 \\
c_1 & 0 & 0 & c_1 & c_2 & c_3-c_1 & 0 \\
0 & 0 & c_1 & c_2 & c_3 & c_4-c_2 & 0
\end{pmatrix}
$$

$$
\times
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
\frac{c_4-c_3}{c_1} & \frac{c_3-c_2}{c_1} & \frac{c_2-c_1}{c_1} & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & \frac{c_2-c_1}{c_1} & \frac{c_3-c_2}{c_1} & \frac{c_4-c_3}{c_1}
\end{pmatrix}.
$$

For n even, the construction slightly changes because the symmetry in the residual with the cross pattern of zeros is different. Let us illustrate it for $n = 6$. The first rank-two correction is the same as for $n = 9$ and we obtain

$$S_6 = \begin{pmatrix} 0 & c_1 & c_2 & c_2 & c_1 & 0 \\ 0 & 0 & c_1 & c_2 & c_2 & c_1 \\ c_1 & 0 & 0 & c_1 & c_2 & c_2 \\ c_2 & c_1 & 0 & 0 & c_1 & c_2 \\ c_2 & c_2 & c_1 & 0 & 0 & c_1 \\ c_1 & c_2 & c_2 & c_1 & 0 & 0 \end{pmatrix} \rightarrow R_6 = \begin{pmatrix} 0 & c_1 & c_2 & c_2 & c_1 & 0 \\ 0 & 0 & c_1 & c_1 & 0 & 0 \\ c_1 & 0 & 0 & 0 & 0 & c_1 \\ c_2 & c_1 & 0 & 0 & c_1 & c_2 \\ c_1 & 0 & 0 & 0 & 0 & c_1 \\ 0 & 0 & c_1 & c_1 & 0 & 0 \end{pmatrix}. \tag{6}$$

However, the fourth row of R_6 is not a copy of the first three. Therefore, we need to keep it: factorizing the following submatrix

$$R_6' = \begin{pmatrix} 0 & c_1 & c_2 \\ 0 & 0 & c_1 \\ c_1 & 0 & 0 \\ c_2 & c_1 & 0 \end{pmatrix}$$

allows to factor R_6 (last three columns and last two rows are duplicates). Since it is a 4-by-3 matrix, we can factor it trivially as $R_6' = R_6' I_3$ and obtain a rank-5 nonnegative factorization of S_6.

In summary,

- At the recursion steps, the factorization of the remaining k-by-l block ($k = l$ or $l + 1$) is computed via a nonnegative rank-two correction and the factorization of its $\lceil k' \rceil$-by-$\lceil \frac{l}{2} \rceil$ upper left block where $k' = \lceil \frac{l}{2} \rceil + 1$ when $k = l$ is even and $k' = \lceil \frac{l}{2} \rceil$ otherwise.
- At the last step, when $k \leq 4$, a trivial factorization is used. Note that there will be four 'basic' cases: 3-by-3 (e.g., for $n = 5, 9$), 4-by-3 (e.g., for $n = 6$), 4-by-4 (e.g., $n = 4, 7$), and 3-by-2 (e.g., for $n = 10$).

In the recursion steps described above, from a large matrix with c columns, a submatrix with $\lceil \frac{c}{2} \rceil$ columns is extracted, and the nonnegative rank of the larger matrix cannot be larger than that of the submatrix plus two (because of the two nonnegative rank-one corrections). This leads to the following result:

Theorem 3. *For any $n \geq 2$, the nonnegative rank of any slack matrix S_n of the regular n-gon is bounded as follows:*

$$\operatorname{rank}_+(S_n) \leq \begin{cases} 2\lceil \log_2(n) \rceil - 1 = 2k - 1 & \text{for} & 2^{k-1} & < & n & \leq 2^{k-1} + 2^{k-2}, \\ 2\lceil \log_2(n) \rceil = 2k & \text{for} & 2^{k-1} + 2^{k-2} < & n & \leq & 2^k. \end{cases} \tag{7}$$

Proof. Let us first assume that the recursion described above is correct, that is, that at each step the number of columns c is decreased to $\lceil \frac{c}{2} \rceil$ while the nonnegative rank is increased by at most 2, unless $c \leq 4$ in which case we use the trivial factorization of rank c. To verify that (7) holds, we observe that the function $\lceil \frac{c}{2} \rceil$ is nondecreasing in c hence it suffices to verify that the upper bound holds for the critical values $2^k, 2^{k-1}+1, 2^{k-1}+2^{k-2}$ and $2^{k-1}+2^{k-2}+1$ for any k. For $n = 2^k$, we check that the recursion divides the number of column by two at each step until the number of columns is equal to four which gives $\mathrm{rank}_+(S_n) \leq 2 \log_2(n)$. For $n = 2^{k-1} + 1$, the number of columns $c = 2^p + 1$ for some p is reduced at each step to $\lceil c/2 \rceil = 2^{p-1} + 1$. After $k - 2$ steps, we get a 3-by-3 matrix which gives $\mathrm{rank}_+(S_n) \leq 2(k-2) + 3 = 2k - 1$. For $n = 2^{k-1} + 2^{k-2}$, after $k - 2$ steps, the number of columns is equal to 3 hence we obtain $\mathrm{rank}_+(S_n) \leq 3 + 2(k-2) = 2k - 1$; the case $n = 2^{k-1} + 2^{k-2} + 1$ is similar to that above.

Let us now prove the recursion. To understand the proof, we encourage the reader to also look at the (short) MATLAB code in Appendix B that constructs the factorizations.[4]

Let B be the k-by-l upper left block of the slack matrix S_n, where $k = l$ or $l + 1$ and $1 \leq k, l \leq n$. Note that, at the first step, $k = l = n$.

Basic step. If $l \leq 4$, B is trivially factorized, that is, $B = BI_l$ where I_l is the l-by-l identity matrix.

Recursion step. If we show that

$$\mathrm{rank}_+(B) \leq 2 + \mathrm{rank}_+(B'),$$

where B' is the k'-by-$\lceil l/2 \rceil$ upper left block of B, where $k' = \lceil k/2 \rceil$ except when $k = l$ is even in which case $k' = \lceil k/2 \rceil + 1 = l/2 + 1$, then the proof will be complete, by recursion (since B' is also a k'-by-l' upper left block of the slack matrix S_n where $l' = \lceil l/2 \rceil$ and $k' = l'$ or $l' + 1$).

Since B is the upper left block of S_n, it is a circulant matrix and has the following form

$$B = \begin{pmatrix} c_0 & c_1 & \cdots & c_{-1+l} \\ c_{-1} & c_0 & \cdots & c_{-2+l} \\ \vdots & \vdots & \cdots & \vdots \\ c_{-k+1} & c_{-k+2} & \cdots & c_{-k+l} \end{pmatrix} = [c_{-i+j}]_{1 \leq i \leq k, 1 \leq j \leq l},$$

where the c_k's are given by (2). The recursion works as follows. First, we subdivide the matrix B into four blocks: (i) the upper left $\lceil l/2 \rceil$-by-$\lfloor l/2 \rfloor$ block, (ii) the upper right $\lceil l/2 \rceil$-by-$\lceil l/2 \rceil$ block, (iii) the lower left $(k - \lceil l/2 \rceil)$-by-$\lfloor l/2 \rfloor$ block, and (iv) the lower right $(k - \lceil l/2 \rceil)$-by-$\lceil l/2 \rceil$ block. (Note that $k - \lceil l/2 \rceil = \lfloor l/2 \rfloor + k - l$ which will be useful later.) Then, we make a nonnegative rank-one correction to the upper right and lower

---

[4] Note that we have numerically checked the correctness of the construction for all $n \leq 10000$.

left blocks so that the off-diagonal entries of B and the entries below are set to zero, that is, all entries (i,j) of B such that $i + j = l + 1$ or $i + j = l + 2$ will be set to zero. (Note that the entries (i,j) of B such that $i = j$ or $i = j + 1$ are already equal to zero.)

Upper right block. Let $p = \lceil l/2 \rceil$ and consider the p-by-p upper right block of B

$$U = \begin{pmatrix} c_{l-p} & c_{l-p+1} & \cdots & c_{l-1} \\ c_{l-p-1} & c_{l-p} & \cdots & c_{l-2} \\ \vdots & \vdots & \cdots & \vdots \\ c_{l-2p+1} & c_{l-2p+2} & \cdots & c_{l-p} \end{pmatrix} = [c_{-i+j}]_{1 \leq i \leq p, l-p+1 \leq j \leq l}$$

$$= [c_{-i+h+l-p}]_{1 \leq i \leq p, 1 \leq h = j - l + p \leq p},$$

from which we remove the matrix $U - [c_{1+p-i-j}]_{1 \leq i \leq p, 1 \leq j \leq p}$ which is equal to

$$\begin{pmatrix} c_{l-p} - c_{p-1} & c_{l-p+1} - c_{p-2} & \cdots & c_{l-1} - c_0 \\ c_{l-p-1} - c_{p-2} & c_{l-p} - c_{p-3} & \cdots & c_{l-2} - c_{-1} \\ \vdots & \vdots & \cdots & \vdots \\ c_{l-2p+1} - c_0 & c_{l-2p+2} - c_{-1} & \cdots & c_{l-p} - c_{-p+1} \end{pmatrix} = [c_{\alpha-i+j} - c_{\beta-i-j}]_{1 \leq i \leq p, 1 \leq j \leq p},$$

where $\alpha = l - p$ and $\beta = 1 + p$. By Lemma 1, that matrix has rank-one. Moreover, it is nonnegative because for all $1 \leq i, j \leq p$

$$c_{l-\lceil l/2 \rceil - i + j} = c_{\lfloor l/2 \rfloor - i + j} \geq c_{1 + \lceil l/2 \rceil - i - j}$$

since $\lfloor l/2 \rfloor + j \geq 1 + \lceil l/2 \rceil - j$ for all j. We obtain

$$[c_{-i+j+l-p} - c_{\alpha-i+j} + c_{\beta-i-j}]_{1 \leq i \leq p, 1 \leq j \leq p} = \begin{pmatrix} c_{p-1} & c_{p-2} & \cdots & c_1 & 0 \\ c_{p-2} & c_{p-3} & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ c_1 & 0 & \cdots & c_{p-4} & c_{p-3} \\ 0 & 0 & \cdots & c_{p-3} & c_{p-2} \end{pmatrix}$$

$$= [c_{p+1-i-j}]_{1 \leq i \leq p, 1 \leq j \leq p}.$$

Lower left block. Let $p = \lfloor l/2 \rfloor$ and $q = p + k - l = k - \lceil l/2 \rceil$ ($= p$ if $k = l$, $= p + 1$ if $k = l + 1$), and consider the q-by-p lower left block of B

$$L = \begin{pmatrix} c_{-k+q} & c_{-k+q+1} & \cdots & c_{-k+q+p-1} \\ \vdots & \vdots & \cdots & \vdots \\ c_{-k+2} & c_{-k+3} & \cdots & c_{-k+p+1} \\ c_{-k+1} & c_{-k+2} & \cdots & c_{-k+p} \end{pmatrix} = [c_{-i+j}]_{k-q+1 \leq i \leq k, 1 \leq j \leq q}$$

$$= [c_{-h-k+q+j}]_{1 \leq h = i - k + q \leq q, 1 \leq j \leq p},$$

from which we remove the matrix $L - [c_{1+p-i-j}]_{1 \leq i \leq q, 1 \leq j \leq p}$ which is equal to

$$\begin{pmatrix} c_{-k+q} - c_{p-1} & c_{-k+q+p} - c_{p-2} & \cdots & c_{-k+q+p-1} - c_0 \\ c_{-k+q-1} - c_{p-2} & c_{-k+q} - c_{p-3} & \cdots & c_{-k+q+p-2} - c_{-1} \\ \vdots & \vdots & \cdots & \vdots \\ c_{-k+2} - c_{p-q+1} & c_{-k+3} - c_{p-q} & \cdots & c_{-k+p+1} - c_{-q+2} \\ c_{-k+1} - c_{p-q} & c_{-k+2} - c_{p-q-1} & \cdots & c_{-k+p} - c_{-q+1} \end{pmatrix}$$

$$= [c_{\alpha-i+j} - c_{\beta-i-j}]_{1\leq i\leq q, 1\leq j\leq p},$$

where $\alpha = -k + q = -\lfloor l/2 \rfloor$ and $\beta = 1 + p = \lfloor l/2 \rfloor + 1$, which can be checked to be non-negative (using the fact that $c_{-k} = c_{k+1}$, we have $c_{\alpha-i+j} = c_{-\alpha+i-j+1} = c_{\lfloor l/2 \rfloor+i-j+1} \geq c_{\lfloor l/2 \rfloor+1-i-j} = c_{\beta-i-j}$), and has rank-one by [Lemma 1]. We obtain

$$[c_{-i-k+q+j} - c_{\alpha-i+j} + c_{\beta-i-j}]_{1\leq i\leq q, 1\leq j\leq p}$$

$$= \begin{pmatrix} c_{p-1} & c_{p-2} & \cdots & c_1 & 0 \\ c_{p-2} & c_{p-3} & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ c_{p-q+1} & c_{p-q}(=0) & \cdots & c_{-q+3} & c_{-q+2} \\ c_{p-q}(=0) & c_{p-q-1} & \cdots & c_{-q+2} & c_{-q+1} \end{pmatrix}.$$

Note that if $k = l$ (that is, $p = q$), then $c_{p-q-1} = 0$, otherwise $k = l + 1$ and $c_{p-q+1} = 0$.

Finally, putting all the blocks together: the untouched upper left and lower right blocks, and the corrected upper right and lower left blocks, we obtain, after a nonnegative rank-two correction of B, the following l-by-l matrix

$$\begin{pmatrix} 0 & c_1 & c_2 & \cdots & c_2 & c_1 & 0 \\ 0 & 0 & c_1 & \cdots & c_1 & 0 & 0 \\ c_1 & 0 & 0 & \cdots & 0 & 0 & c_1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ c_2 & c_1 & 0 & \cdots & 0 & c_1 & c_2 \\ c_1 & 0 & 0 & \cdots & 0 & 0 & c_1 \\ 0 & 0 & c_1 & \cdots & c_1 & 0 & 0 \end{pmatrix}$$

to which the following row

$$\begin{pmatrix} 0 & c_1 & c_2 & \cdots & c_2 & c_1 & 0 \end{pmatrix}$$

has to be added when $k = l + 1$. That matrix has the following properties

- every column is repeated twice except the middle one when l is odd – more precisely, the jth and $(l - j + 1)$th columns are identical for $1 \leq j \leq \lfloor l/2 \rfloor$ –, and
- every row is repeated twice except (i) the first one when $k = l$, (ii) the $(l/2 + 1)$th when $k = l$ is even, (ii) the middle one when $k = l + 1$ is odd – more precisely, the $(i+s)$th and $(k-i+1)$th rows are identical for $1 \leq i \leq \lfloor k/2 \rfloor$, and $s = 0$ for $k = l+1$ and $s = 1$ for $k = l$.

This concludes the recursion step, hence the proof. □

A MATLAB code that generates the slack matrices of regular n-gons and constructs the nonnegative factorizations described above for any n is available from

https://sites.google.com/site/exactnmf/regularngons

Tightness of the bound   It has to be pointed out that our inspiration for constructing the nonnegative factorizations used in Theorem 3 came from factorizations computed by our numerical solver [23] available on https://sites.google.com/site/exactnmf/.

Moreover, for n up to 78, the heuristic algorithm developed in [23] was systematically able to find a factorization with a rank matching the bound of Theorem 3, and never with a smaller rank. This suggests that our upper bound is tight, at least for small n.

5. Conclusion

In this paper, we have first proposed a new lower bound for the rectangle covering number of the slack matrix of any n-gon, using a generalization of Sperner's theorem; see Theorem 2 and Corollary 3. We hope that this idea will lead to new lower bounds for other types of nonnegative matrices.

Then, we proposed a purely algebraic proof for an upper bound on the extension complexity of regular n-gons, based on explicit nonnegative factorizations of the corresponding slack matrices; see Theorem 3. This bound slightly improves upon the previously best known upper bound from [9] (our improvement essentially comes from improving the base case, but we provided a new algebraic proof), and allows us to close the gap with the best known lower bound for several n-gons ($9 \leq n \leq 13$, $21 \leq n \leq 24$; see Fig. 2). However, for most n-gons (precisely, for $n = 14$, $17 \leq n \leq 20$, $25 \leq n \leq 30$ and $n \geq 33$), a gap subsists between the best known lower and upper bounds. Hence it is a direction for further research to seek improvements to these bounds, in order to determine the extension complexity of these regular n-gons. Our numerical results suggest that the way to go would be to improve the lower bounds, since our upper bound appears to be tight, at least for small n.

Acknowledgements

Appendix A. Proof for Theorem 2

The solution $k^* = \lfloor r/2 \rfloor$ and $z^* = 1$ is optimal for

$$\min_{k \geq 1, z \geq 1, k+z \leq r} k!(r-k)! + (k+z)!(r-k-z)! - 2k!z!(r-k-z)!.$$

Proof. Let us observe the following

- the first (resp. second) term is decreasing when k (resp. $k+z$) gets closer to $r/2$;
- the last term is strictly increasing in z hence being minimized in $z = 1$;
- $f(k,z) = f(r-k-z, z)$. (Note that this implies that, for r even, $k^* = r/2 - 1$ is also optimal.)

The first two observations imply that, at optimality, the case $z \geq 2$ and $k + z \geq \lfloor r/2 \rfloor + 1$ is not possible, otherwise we would decrease the objective function by decreasing z. In other words, either $z^* = 1$ or $k + z \leq \lfloor r/2 \rfloor$.

Case 1: $z^* = 1$. Since $f(k,1) = f(r-k-1, 1)$, we can assume w.l.o.g. that $k \geq \lfloor r/2 \rfloor$ since either k or $r - k - 1$ is larger than $\lfloor r/2 \rfloor$. Showing that $f(k,1)$ is increasing for $\lfloor r/2 \rfloor \leq k \leq r - 1$, that is, that $f(k,1) \leq f(k+1,1)$ for $k + 1 \leq r - 1$ will prove the result:

$$k!(r-k)! + (k+1)!(r-k-1)! - 2k!(r-k-1)!$$
$$\leq (k+1)!(r-k-1)! + (k+2)!(r-k-2)! - 2(k+1)!(r-k-2)!$$

$$\Longleftrightarrow$$

$$k!(r-k)! - 2k!(r-k-1)! \leq (k+2)!(r-k-2)! - 2(k+1)!(r-k-2)!.$$

Dividing by $k!$ and $(r-k-2)!$,

$$(r-k)(r-k-1) - 2(r-k-1) \leq (k+2)(k+1) - 2(k+1)$$

which is equivalent to

$$r^2 - 3r + 2 \leq 2k(r-1).$$

Since $k \geq \lfloor r/2 \rfloor$, $2k \geq r - 1$ hence the above inequality would be implied by

$$r^2 - 3r + 2 \leq (r-1)^2 = r^2 - 2r + 1 \iff r \geq 1.$$

Case 2: $k+z \leq \lfloor r/2 \rfloor$. We have $k' = r - k - z \geq \lfloor r/2 \rfloor$ hence we can reduce this case to the case $k \geq \lfloor r/2 \rfloor$ without loss of generality, since $f(k,z) = f(r-k-z, z)$. For $k \geq \lfloor r/2 \rfloor$, it is clear that $z^* = 1$ is optimal (since in that case the last two terms increase with z) so that this case reduces to Case 1 when $z^* = 1$. $\square$

Appendix B. Code for the nonnegative factorization of slack matrices of regular n-gons

```
% Rank r nonnegative factorization of the slack matrix S_n=UV of the regular
% n-gon generated by the function slack.m, r being equal to
% 2k-1  for  2^{k-1}           <   n   <=  2^{k-1}+2^{k-2} , and
% 2 k   for  2^{k-1}+2^{k-2} <   n   <=  2^{k} .
%
% See A. Vandaele, N. Gillis and F. Glineur,
% "On the Linear Extension Complexity of Regular n-gons", arXiv, 2015.
% If you use the code, please cite the paper.
% See also https://sites.google.com/site/exactnmf/regularngons
%
% This version uses the matrix S as an input with O(n^2) operations,
% hence is computationally less efficient as factorization.m which
% only requires O(n log(n)).
% However, it is more intuitive to understand the construction and follows
% the proof of the paper above more closely.

function [U,V,R] = NonnegFactoRegnGon(S)

[m,n] = size(S);
if n <= 4 % trivial factorization
    U = S;
    V = eye(n);
    R = S;
else n > 4 % non-trivial factorizations
    % Step 1: Create the cross pattern of zeros removing a nonnegative
    % rank-two factor
    [U,V,R] = offdiag_zeros(S);
    % Step 2: Extract the upper left block that has the same nonnegative
    % rank as the full residual R (because of symmetry/redundancy)
    k1 = ceil(m/2);
    if k1 == m/2 && m == n % When m is even and m == n
        k1 = k1+1;
    end
    k2 = ceil(n/2);
    % Step 3: Factor the upper left block using recursion
    [Ur,Vr] = NonnegFactoRegnGon(S(1:k1,1:k2));
    % Step 4: Put everything together using the symmetry
    r = size(Ur,2);
    U = [U zeros(m,r)];
    V = [V; zeros(r,n)];
    % Factor V
    V(3:end,1:k2) = Vr;
    for i = n : -1 : k2+1
        V(3:end,i) = V(3:end,n-i+1);
    end
    % Factor U
    U(1:k1,3:end) = Ur;
    if m == n % Case 1: R(k1,:)==R(k1+1,:), symmetry is 'perfect'
        p = 1;
    elseif m == n+1 % Case 2: R(k1-1,:)==R(k1+1,:), symmetry is shifted by one
        p = 0;
    end
    for i = 1 : m-k1
        U(k1+i,3:end) = U(k1-i+p,3:end);
```

```
      end
end

% Add zeros on off-diagonal entries of matrix S using a rank-two
% correction. The first rank-one factor puts zero entries on the lower left
% block, the second on the upper right block.

function [U,V,R] = offdiag_zeros(S)

[m,n] = size(S);
U = zeros(m,2);
V = zeros(2,n);
k2 = floor(n/2);
% Lower left block
if m == n  % zeros below the diagonal (starting from the lower left)
    k1 = ceil(n/2);
    U(m,1) = S(m,1);
elseif m == n+1  % zeros above the diagonal (starting from the lower left)
    k1 = floor(m/2);
    U([m-1 m],1) = S([m-1 m],1);
else
    error('The matrix should be n-by-n or n+1-by-n');
end
V(1,1) = 1;
for i = 2 : k2
    V(1,i)    = S(n-i+2,i) / U(n-i+2,1);
    U(n-i+1,1) = S(n-i+1,i)/ V(1,i);
end
% Upper right block: zeros below the diagonal
% (starting from the upper right)
V(2,n) = 1; U(1,2) = S(1,n);
for i = 2 : k1
    U(i,2)     = S(i,n-i+2) /  V(2,n-i+2);
    V(2,n-i+1)= S(i,n-i+1) / U(i,2);
end
% Residual with the pattern of zeros like a cross
R = S - U*V;
```

The code is available from https://sites.google.com/site/exactnmf/regularngons.

References

[1] C. Barefoot, K.A.S. Hefner, K.F. Jones, J.R. Lundgren, Biclique covers of the complements of cycles and paths in a digraph, Congr. Numer. 53 (1986) 133–146.
[2] L.B. Beasley, T.J. Laffey, Real rank versus nonnegative rank, Linear Algebra Appl. 431 (12) (2009) 2330–2335.
[3] A. Ben-Tal, A. Nemirovski, On polyhedral approximations of the second-order cone, Math. Oper. Res. 26 (2) (2001) 193–205.
[4] D. de Caen, D.A. Gregory, N.J. Pullman, The boolean rank of zero-one matrices, in: Proc. 3rd Caribbean Conference on Combinatorics and Computing, 1981, pp. 169–173.
[5] H. Fawzi, P.A. Parrilo, Lower bounds on nonnegative rank via nonnegative nuclear norms, Math. Program. 153 (1) (2015) 41–66.
[6] H. Fawzi, P.A. Parrilo, Self-scaled bounds for atomic cone ranks: applications to nonnegative rank and cp-rank, Math. Program. (2015).
[7] S. Fiorini, V. Kaibel, K. Pashkovich, D.O. Theis, Combinatorial bounds on nonnegative rank and extended formulations, Discrete Math. 313 (1) (2013) 67–83.

[8] S. Fiorini, S. Massar, S. Pokutta, H.R. Tiwary, R. de Wolf, Linear vs. semidefinite extended formulations: exponential separation and strong lower bounds, in: Proceedings of the Forty-Fourth Annual ACM Symposium on Theory of Computing, ACM, 2012, pp. 95–106.

[9] S. Fiorini, T. Rothvoss, H.R. Tiwary, Extended formulations for polygons, Discrete Comput. Geom. 48 (3) (2012) 658–668.

[10] N. Gillis, F. Glineur, On the geometric interpretation of the nonnegative rank, Linear Algebra Appl. 437 (11) (2012) 2685–2712.

[11] F. Glineur, Computational experiments with a linear approximation of second order cone optimization, Image Technical Report 0001, Service de Mathématique et de Recherche Opérationnelle, Faculté Polytechnique de Mons, 2000.

[12] M.X. Goemans, Smallest compact formulation for the permutahedron, Math. Program. 153 (1) (2015) 5–11, http://math.mit.edu/~goemans/PAPERS/permutahedron.pdf.

[13] J. Gouveia, R. Grappe, V. Kaibel, K. Pashkovich, R.Z. Robinson, R.R. Thomas, Which nonnegative matrices are slack matrices?, Linear Algebra Appl. 439 (10) (2013) 2921–2933.

[14] J. Gouveia, P.A. Parrilo, R.R. Thomas, Lifts of convex sets and cone factorizations, Math. Oper. Res. 38 (2) (2013) 248–264.

[15] V. Kaibel, K. Pashkovich, Constructing extended formulations from reflection relations, in: M. Jünger, G. Reinelt (Eds.), Facets of Combinatorial Optimization, Springer, Berlin, Heidelberg, 2013, pp. 77–100.

[16] Kaibel Volker, Kanstantsin Pashkovich, Constructing extended formulations from reflection relations, in: Integer Programming and Combinatorial Optimization, Springer, 2011, pp. 287–300.

[17] D. Lubell, A short proof of Sperner's lemma, J. Combin. Theory 1 (2) (1966) 299.

[18] P. McMullen, The maximum numbers of faces of a convex polytope, Mathematika 17 (02) (1970) 179–184.

[19] M. Oelze, A. Vandaele, S. Weltge, Computing the extension complexities of all 4-dimensional 0/1-polytopes, arXiv:1406.4895, 2014.

[20] T. Rothvoss, The matching polytope has exponential extension complexity, in: Proceedings of the 46th ACM Symposium on Theory of Computing, STOC '14, 2014, pp. 263–272.

[21] Y. Shitov, An upper bound for nonnegative rank, J. Combin. Theory Ser. A 122 (2014) 126–132.

[22] E. Sperner, Ein satz über untermengen einer endlichen menge, Math. Z. 27 (1) (1928) 544–548.

[23] Arnaud Vandaele, Nicolas Gillis, François Glineur, Daniel Tuyttens, Heuristics for exact nonnegative matrix factorization, J. Global Optim. 65 (2) (2016) 369–400.

[24] M. Yannakakis, Expressing combinatorial optimization problems by linear programs, J. Comput. System Sci. 43 (3) (1991) 441–466.

[25] G.M. Ziegler, Lectures on Polytopes, Springer-Verlag, 1995.